A RECURRENCE FORMULA FOR VISCOELASTIC CONSTITUTIVE EQUATIONS

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CONSTITUTIVE EQUATIONS*

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INTRODUCTION

The viscoelastic constitutive equations are generally represented by integral equations with kernels. These kernels are functions of current time, an integration limit of the hereditary integral. Therefore, the values of these kernels change as the time increases and the integral must be evaluated from time equals zero to the current time for every increment of time. Thus, as time increases, the required computing time becomes longer and longer. Furthermore, all physical values from time equals zero to the current time must be stored for later evaluations of these integrals. Additionally, for finite deformation viscoelastic problems, the constitutive equation is an integral part of the equilibrium equations that result in a set of nonlinear differential-integral equations. These equations usually can only be solved numerically and iteratively. Hence, computing time and data storage are the main concerns in solving finite deformation viscoelastic problems. The main object of this paper is to develop a method that saves both computing time and data storage in evaluating these integral equations.

In this paper, the constitutive integral equations are written in the incremental form and a recurrence formula is obtained. This formula has been previously developed for the kernel represented by the exponential form only[1]. Here a new recurrence formula has been obtained. The new

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formulation does not restrict the form of the kernel. The kernel may be of
any differentiable functions. With the recurrence formula, the value of a
hereditary integral at current time step depends only upon the value at the
previous time step. Thus, no data storage is required and computing time is
reduced.

As an example, the constitutive equation, developed by Christensen[2],
for large deformation viscoelastic problems, is used. The recurrence formulas
are obtained explicitly for the relaxation functions that are written in terms
of either the exponential law or the power law.

FORMULATION

The constitutive equation for viscoelastic materials can often be
described by the Volterra integral equation of the first kind

$$f(x) = \phi(x) - \lambda \int_O^x k(x,y) \, \phi(y) dy \qquad (1)$$

where $\phi(x)$ is usually an input function and $f(x)$ is the responding function of
the material. The kernel $k(x,y)$ satisfies the condition

$$k(x,y) = 0 \qquad \text{if } y > x, \qquad (2)$$

and is further assumed to be continuous and differentiable in this paper.

If x_{n+1} denotes the value of x at the n+1 increments, i.e.,

$$x_{n+1} = x_n + \Delta x \qquad (3)$$

then equation (1) can be written as

$$f(x_{n+1}) = \phi(x_{n+1}) - \lambda \, I_{n+1} \qquad (4)$$

where

$$I_{n+1} = \int_O^{x_{n+1}} k(x_{n+1},y) \, \phi(y) \, dy \qquad (5)$$

The value of Δx is a small quantity but it need not be a constant increment. With the assumption that the kernel is continuous and differentiable and with Taylor's series expansion, one obtaines the following expression for $k(x_{n+1}, y)$

$$k(x_{n+1}, y) = k(x_n, y) + \Delta x\, k^1(x_n, y) + \cdots + \frac{(\Delta x)^{m-1}}{(m-1)!} k^{m-1}(x_n, y) + R_m \qquad (6)$$

where, R_m is the remainder in Taylor's sereis

$$R_m = \frac{(\Delta x)^m}{m!} k^m(\xi) \qquad x < \xi < x + \Delta x \qquad (7)$$

The $k^r(x, y)$ in the above equations denotes the r th order differentiation of $k(x, y)$ with respect to x. Substituting equation (6) into equation (5) and dividing the integral I_{n+1} into two parts with limits from 0 to x_n and from x_n to x_{n+1}, one obtains

$$I_{n+1} = I_n + \Delta x\, I_n^1 + \cdots + \frac{(\Delta x)^{m-1}}{(m-1)!} I_n^{m-1} + \cdots +$$

$$\int_{x_n}^{x_n + \Delta x} k(x_n + \Delta x, y)\, \phi(y)\, dy \qquad (8)$$

where

$$I_n^r = \int_o^{x_n} k^r(x_n, y)\, \phi(y)\, dy \qquad (9)$$

With the mean-valued theorem, the following recurrence formula is obtained for the integral

$$I_{n+1} = I_n + \Delta x\, I_n^1 + \cdots + \frac{(\Delta x)^{m-1}}{(m-1)!} I_n^{m-1} + \cdots$$

$$+ k(x_{n+1}, x_n + \frac{1}{2}\Delta x) \left[\phi(x_{n+}) - \phi(x_n) \right] \Delta x \qquad (10)$$

The integrals I_n^r are evaluated at the previous step. The recurrence formula for I_{n+1}^r can be obtained similar to I_{n+1}, i.e.

$$I_{n+1}^r = I_n^r + \Delta x \ I_n^{r+1} + \cdots + \frac{(\Delta x)^{m-r-1}}{(m-r-1)!} \ I_n^{m-1} + \cdots$$

$$+ k^r(x_{n+1}, x_n + \frac{1}{2} \Delta x) \left[\phi(x_{n+1}) - \phi(x_n) \right] \Delta x \qquad (11)$$

The initial conditions are

$$I_o = I_o^1 = \cdots = I_o^{m-1} = \cdots = 0 \qquad (12)$$

Substituting equations (10-12) into equation (4) for I_{n+1}, the value for $f(x)$ at the current n+1 th step is obtained. These values depend upon the values of the integral and its derivatives at the previous n th step only.

SPECIAL CASES

The application of the recurrence formula for the Volterra integral equation to viscoelastic constitutive equations is mentioned in this paragraph. The constitutive equation, developed by Christensen[2] and rewritten here for reference, is used for illustration:

$$\sigma_{ij}(t) = - p \ \delta_{ij} + x_{i,K}(t) \ x_{j,L}(t) \left[g_o \ \delta_{KL} + A_{KL}(t) \right] \qquad (13)$$

and

$$A_{KL}(t) = \int_o^t g_1(t-\tau) \ \frac{\partial E_{KL}(\tau)}{\partial \tau} \ d\tau \qquad (14)$$

In equations (13) and (14), σ_{ij} is the Cauchy stress, p is the pressure, $x_{i,K}$ denotes the displacement gradient, δ_{KL} is the Kronecker delta and E_{KL} are the components of the Green strain tensor. There are two material property functions g_0 and g_1. The g_0 term is the contribution of the kinetic theory of rubber elasticity; it is related closely to the material constant used in neo-

Hookean constitutive equation in finite elasticity. The $g_1(t)$ term is the relaxation function; it can usually be written in terms of the exponential law or the power law.

When the relaxation function is written in terms of the exponential law

$$g_1(t) = e^{-t} \tag{15}$$

then,

$$A_{KL}(t_n) = - A_{KL}^1(t_n) = A_{KL}^2(t_n) = \bullet\bullet\bullet \tag{16}$$

Equation (10) reduces to

$$A_{KL}(t_{n+1}) = \left[1-\Delta t + \frac{(\Delta t)^2}{2!} - \frac{(\Delta t)}{3!} + \bullet\bullet\bullet\right] A_{KL}(t_n)$$

$$- e^{-\Delta t/2} \left[E_{KL}(t_{n+1}) - E_{KL}(t_n)\right] \tag{17}$$

With the initial condition that $A_{KL}(0)=0$, the constitutive equation (13) can be written in the recurrence form

$$\sigma_{ij}(t_{n+1}) = - p\delta_{ij} + x_{i,K}(t_{n+1}) \, x_{j,L}(t_{n+1}) \left\{ g_0 \, \delta_{KL} \right.$$

$$\left. + e^{-\Delta t} A_{KL}(t_n) + e^{-\frac{\Delta t}{2}} \left[E_{KL}(t_{n+1}) - E_{KL}(t_n)\right] \right\} \tag{18}$$

The Cauchy stresses in (17) at time t_{n+1} depend only on the values evaluated at t_n and the deformation gradients at t_n and t_{n+1}.

When the relaxation function is written in terms of the power law

$$g_1(t) = t^\alpha \tag{19}$$

the r th derivative of $g_1(t)$ with respect to t is

$$g_1^r(t) = \alpha \; (\alpha-1) \cdots (\alpha-r+1) \; t^{(\alpha-r)} \tag{20}$$

Equation (10) reduces to

$$A_{KL}(t_{n+1}) = A_{KL}(t_n) + \Delta t \; A_{KL}^1(t_n) + \cdots \tag{21}$$

$$+ \frac{(\Delta t)^{m-1}}{(m-1)!} A_{KL}^{m-1}(t_n) \cdots + \left(\frac{\Delta t}{2}\right)^\alpha \left[E_{KL}(t_{n+1}) - E_{KL}(t_n)\right]$$

The recurrence formulas for the derivatives in equation (21) are

$$A_{KL}^r(t_{n+1}) = A_{KL}^r(t_n) + \Delta t \; A_{KL}^{r+1}(t_n) + \cdots + \frac{(\Delta t)^{m-r-1}}{(m-r-1)!} A_{KL}^{m-1}(t_n) + \cdots \tag{22}$$

$$+ \alpha(\alpha-1) \cdots (\alpha-r+1) \; (\frac{\Delta t}{2})^{(\alpha-r)} \left[E_{KL}(t_{n+1}) - E_{KL}(t_n)\right]$$

and the initial conditions are

$$A_{KL}(0) = A_{KL}^1(0) = \cdots = A_{KL}^{m-1}(0) = \cdots = 0 \tag{23}$$

With equations (21-23), the current values of the integral in equation (13) depend on the previous time step only. The recurrence formula for viscoelastic constitutive equations, with relaxation functions represented by the power law, is again obtained.

DISCUSSION

There are many representations for the constitutive equation for viscoelastic materials. Only the one developed by Christensen is presented in this paper as a special case. However, the general formulation should be applicable to all convolution integrals. Furthermore, the forms of the relaxation or creep functions are not limited to the exponential or power laws as presented in this paper. The general recurrence formula can be used for any relaxation and creep functions that are continuous and differentiable. The viscoelastic problems, combined with the recurrence formula, can be solved just like the elastic problems for each time step.

REFERENCES

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